10/11/22 MATH4030 Tutorial

Reminders:

- Assignment 5 due Tuesday 11:59pm (15/11).

Recall Def: d on M_1 is given by $d(x,y) = \inf\{ \ell(\alpha),\ \text{s.t. } \alpha(0)=x,\ \alpha(1)=y\}$.

- $\varphi: M_1 \to M_2$ diffeomorphism, φ is an isometry if for any $p \in M_1$, the map $d\varphi_p : T_pM_1 \to T_{\varphi(p)}M_2$ is an isometry of inner product spaces, i.e. for any $w_1, w_2 \in T_pM_1$

$$\underbrace{\langle w_1, w_2 \rangle_p}_{\text{inner product in } T_pM_1} = \underbrace{\langle d\varphi_p(w_1), d\varphi_p(w_2) \rangle_{\varphi(p)}}_{\text{inner product } T_{\varphi(p)}M_2}.$$

By the polarization identity: $|v+w|^2 = \langle v+w, v+w \rangle \overset{\langle v,v\rangle + \langle v,w\rangle + \langle w,v\rangle + \langle w,w\rangle}{=} |v|^2 + 2\langle v,w\rangle + |w|^2$

$$\Leftrightarrow \langle v,w \rangle = \frac{1}{2}\left(|v+w|^2 - |v|^2 - |w|^2 \right).$$

if φ is an isometry, $\forall p \in M$, $v \in T_pM_1$, $|d\varphi_p(v)| = |v|$.

<u>Problem 1</u> dC 4-2 Q3.

Show that a diffeomorphism $\varphi: M_1 \to M_2$ is an isometry iff the arclength of any parametrized curve in M_1 is equal to the arclength of the image curve under φ.

Pf: $\Leftarrow$: Suppose φ is an isometry. Let $\alpha: [a,b] \to M_1$ be a parametrized curve.

$$l_a^b(\varphi(\alpha)) = \int_a^b |(\varphi\circ\alpha)'(t)|\,dt = \int_a^b (\langle \varphi\circ\alpha'(t), (\varphi\circ\alpha)'(t)\rangle)^{1/2}\,dt$$

$$= \int_a^b (\langle \alpha'(t), \alpha'(t)\rangle)^{1/2}\,dt = l_a^b(\alpha).$$

$\Rightarrow$: Now suppose $l_a^b(\alpha) = l_a^b(\varphi(\alpha))$. Let $p \in M_1$, $v \in T_pM_1$. Suppose for contradiction that $|d\varphi_p(v)| \neq |v|$, WLOG, $|d\varphi_p(v)| > |v|$. Take α be a curve s.t. $\alpha(0) = p$ $\alpha'(0) = v$. Then by smoothness of α, there is a nbhd of 0, $(-\varepsilon, \varepsilon)$ s.t. $|d\varphi_p(\alpha'(t))| > |\alpha'(t)|$, but then that means

$l^{\varepsilon}_{-\varepsilon}(\alpha(t)) < l^{\varepsilon}_{-\varepsilon}(\varphi(\alpha(t)))$ which is a contradiction. //

<u>Recall Def</u>: $\varphi: M_1 \to M_2$ diffeomorphism is <u>conformal</u>, if $\forall p \in M_1$, $w_1, w_2 \in T_pM_1$, we have

$$\langle d\varphi_p(w_1), d\varphi_p(w_2)\rangle_{\varphi(p)} = \lambda^2 \langle w_1, w_2\rangle_p$$

for a nowhere zero smooth function λ on M_1.

- isometry $\Rightarrow$ conformal & $\lambda \equiv 1$.

Problem 2 dC 4-2 Q18

a diffeomorphism $\varphi: M_1 \to M_2$ is area preserving if $A(R) = A(\varphi(R))$ for any region $R \subset M_1$.
Prove that if φ is area preserving and conformal, then φ is an isometry.
Note: You may use the fact that if $X(u,v)$ parametrizes M_1,
$\bar{X}(u,v)$ parametrizes M_2,

then $$d\varphi_p(X_u) = \bar{X}_u$$
$$d\varphi_p(X_v) = \bar{X}_v. \quad (*)$$

Pf: Using the fact above, and by conformality, we have

$$\bar{E} = \langle \bar{X}_u, \bar{X}_u \rangle = \langle d\varphi_p(X_u), d\varphi_p(X_u) \rangle = \lambda^2 \langle X_u, X_u \rangle = \lambda^2 E.$$

$$\bar{F} = \lambda^2 F$$

$$\bar{G} = \lambda^2 G.$$

Then we have $\sqrt{\bar{E}\bar{G} - \bar{F}^2} = \lambda^2 \sqrt{EG - F^2}$

$$\int_{X^{-1}(R)} \sqrt{EG-F^2}\,du\,dv \underset{\substack{\text{change of}\\ \text{variables}}}{=} \int_{\bar{X}^{-1}(\varphi(R))} \sqrt{\bar{E}\bar{G}-\bar{F}^2}\,du\,dv = A(\varphi(R)) = A(R) = \int_{X^{-1}(R)} \sqrt{EG-F^2}\,du\,dv.$$

$$\| $$

$$\int_{X^{-1}(R)} \lambda^2 \sqrt{EG-F^2}\,du\,dv. \quad \Rightarrow \text{forces } \lambda \equiv 1 \text{, hence } \varphi \text{ is an isometry.}$$

(*): Since φ is a orientation-preserving diffeomorphism, we know that $d\varphi_p$ is an isomorphism of vector spaces:

$$\varphi \circ \varphi^{-1} = id_{M_2} \Rightarrow d(\varphi \circ \varphi^{-1})_{\varphi(p)} = d(id_{M_2}) = id_{T_{\varphi(p)}M_2}$$

$$\| $$

$$d\varphi_p \circ (d\varphi^{-1})_{\varphi(p)}$$

Same argument for $\varphi^{-1} \circ \varphi = id_{M_1}$ to obtain an inverse for $d\varphi_p$.

Fact from linear algebra: isomorphism of vector spaces maps bases to bases.

$$X_u \mapsto \bar{X}_u$$
$$X_v \mapsto \bar{X}_v.$$